ON NONSTANDARD VISUALIZATION

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Abstract. We discuss the nonstandard (non-Euclidean, four-dimensional, of variable dimension, and with two-edged space placement) visualizations, their neurophysiological and philosophical possibilities, and ways to realize them.

From about fifteen years the second author is involved into graph theory, mainly, into visual graph representation. It has a limited number of methods and his experience shows that it is often hard to obtain a good visual representation of given graphs, especially of complicated graphs coming from real life. This suggested him to consider nonstandard visualization, which, on the other hand, should be one of branches of his project of the investigation of the mind space. The main purpose of this paper is to push interest to nonstandard visualisation, to motivate people to deal with it, to advance it.

Related work. Nonstandard visualization is already used in art (see, for instance, [31]) and its possibilities can be illustrated by drawings by the well-known Dutch artist Maurits Cornelis Escher (1898–1971).[1] Namely, there are drawings of objects which are non-Euclidean (see "Ascending and descending", "Balcony" (Fig. 1), "Belvedere", "Print Gallery", and "Waterfall" (Fig. 2)), (see also "Butterflies", "Circle Fish", "Circle limit II", "Circle limit III" (Fig. 3), "Lizards", "Smaller and smaller", "Path of life II", "Snakes", and "Whirlpools", see also [14]); have two-edged space placement (see "Another World 2", "Convex and concave" (Fig. 4), "Cube with magic ribbon", and "Relativity" (Fig. 5)), and even have variable dimension (see "Cycle", "Day and night", "Doric", "Drawing hands", "Encounter", "Metamorphosis III" (Fig. 6), "Mirror", "Predestination", and "Reptiles" (Fig. 7)).

Key words and phrases. nonstandard visualization, 4D, virtual reality, Thurston's geometrization conjecture, hyperbolic space, Maurits Cornelis Escher, Charles Howard Hinton, Piotr Uspenskiy, the Tibetan Book of the Dead, Dharmakaya, bardo of dharmata.

[1]The referenced drawings are shown at the next pages. The remained listed drawings can be downloaded from `mega.nz/file/N8B3ET5C#jRqkjW-_2lBHL3AG8SDAlHYmkOP46AQ7l3ldO-Xfodk`.

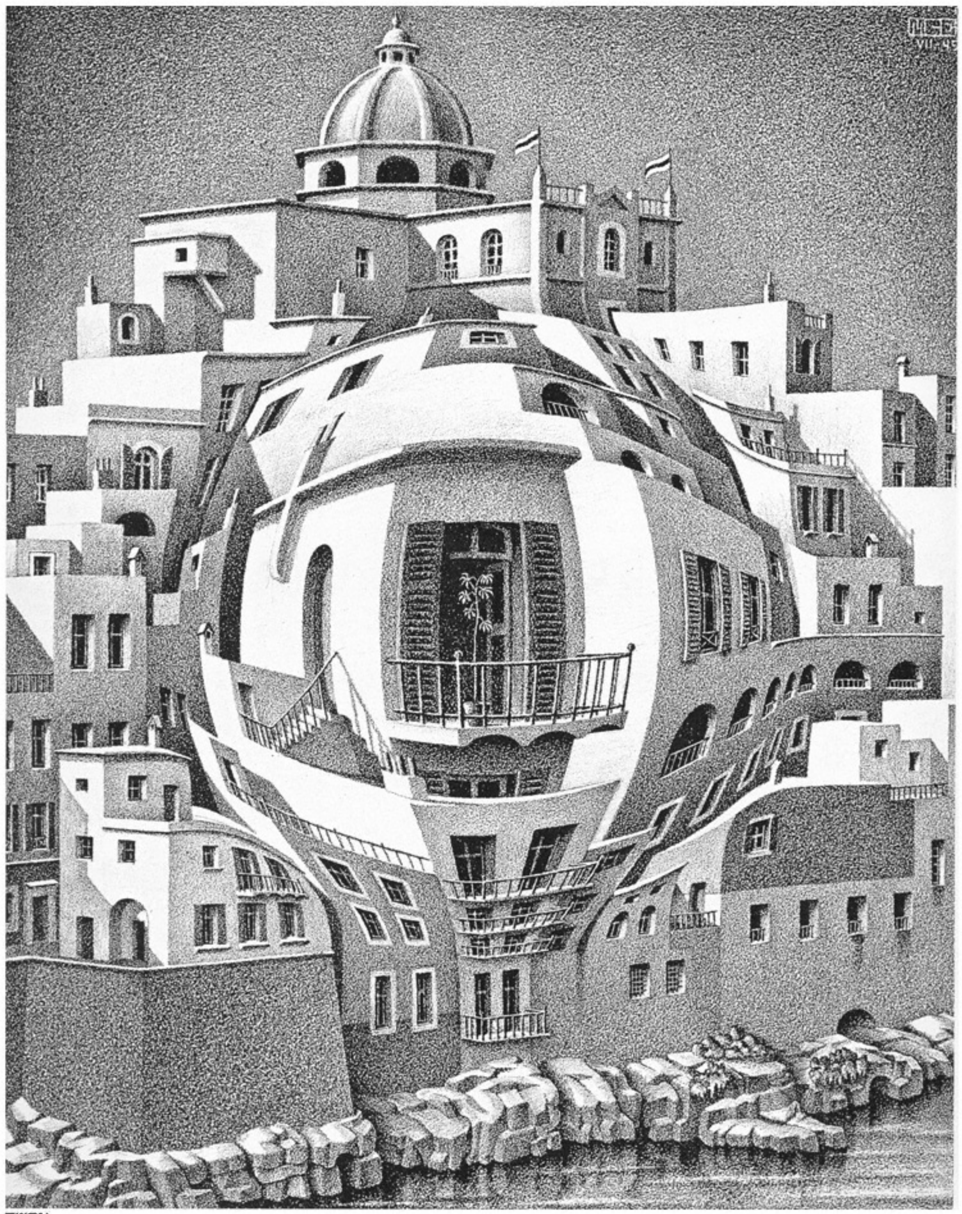


Figure 1. The non-Euclidean visualisation in "Balcony" by M.C. Escher.

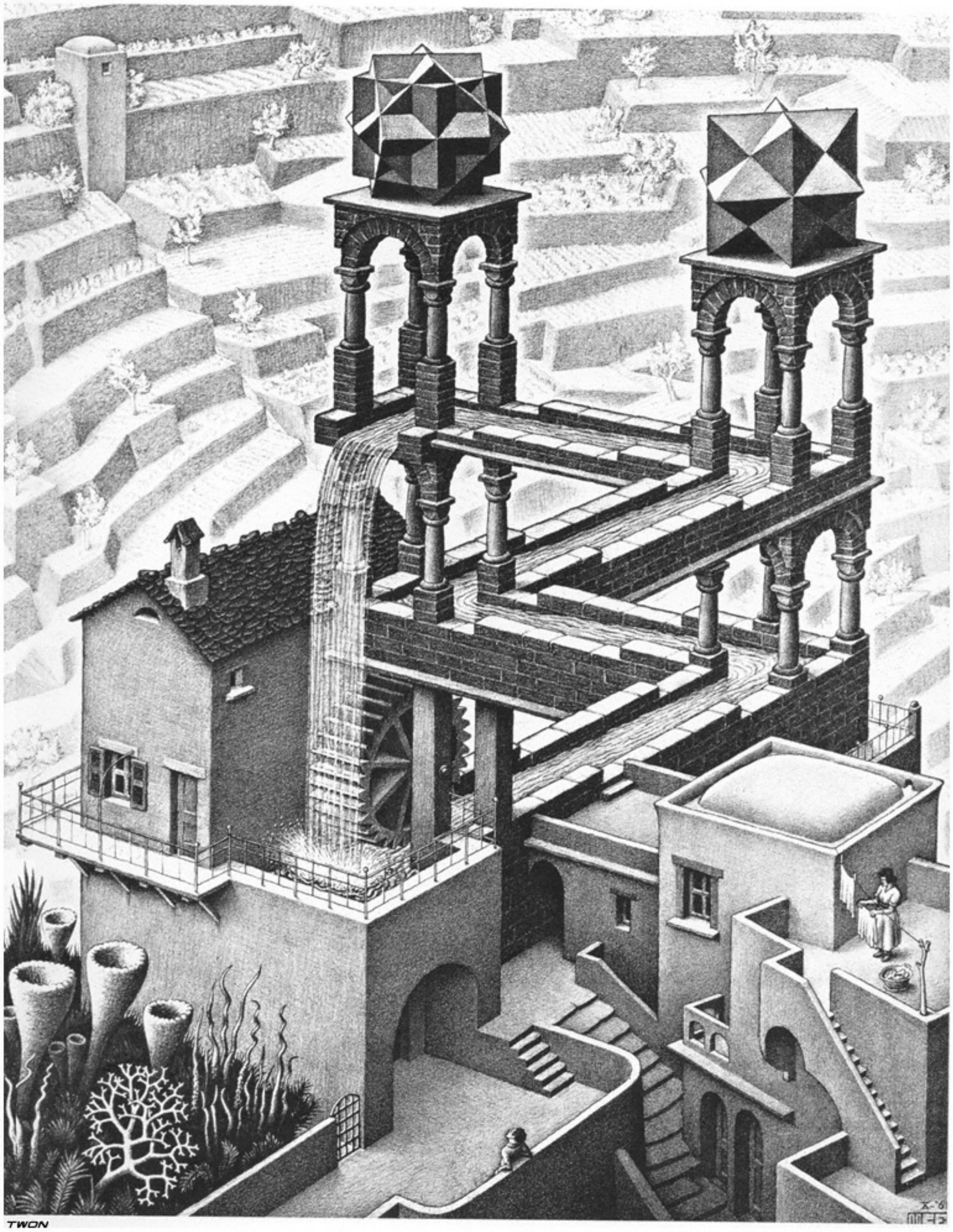


FIGURE 2. The non-Euclidean visualisation in "Waterfall" by M.C. Escher.

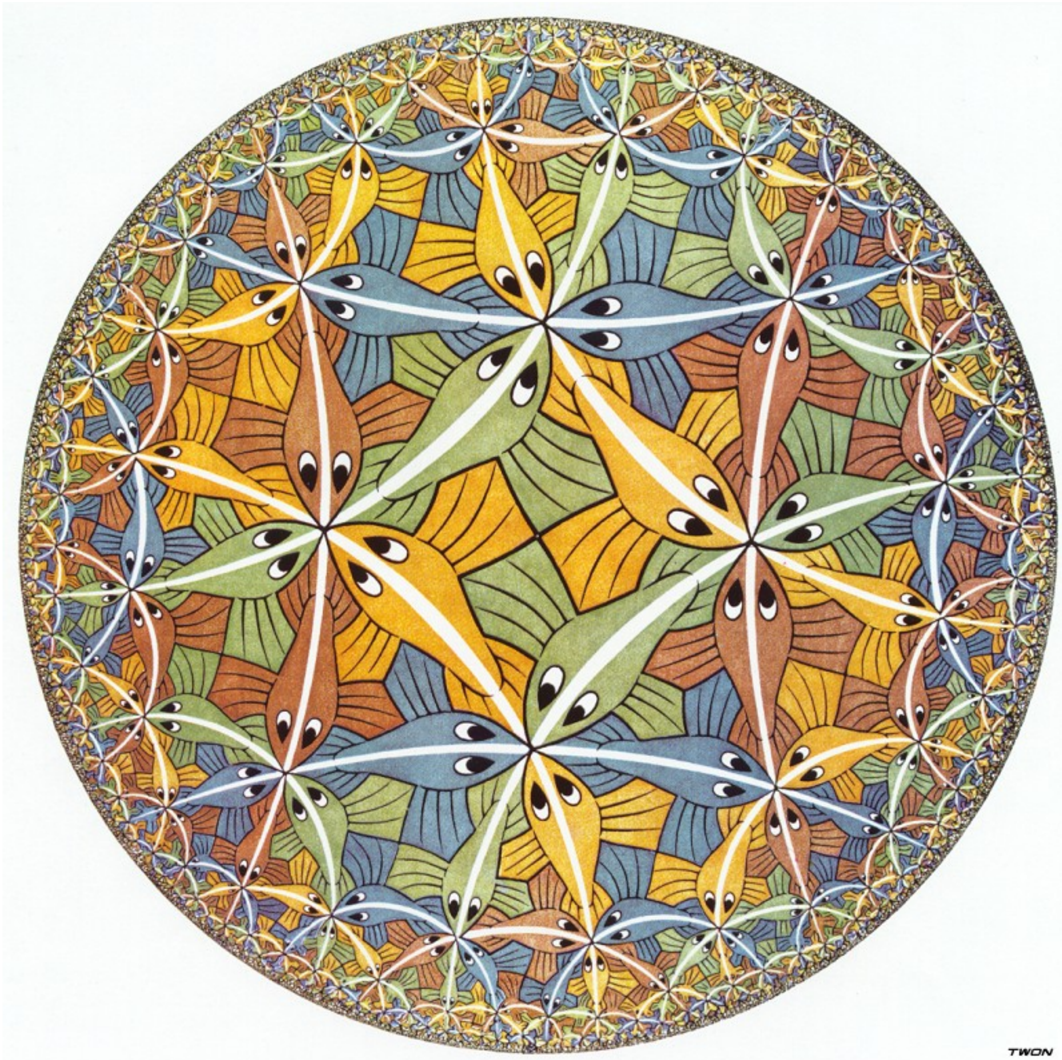


FIGURE 3. The non-Euclidean visualisation in “Circle limit III” by M.C. Escher.

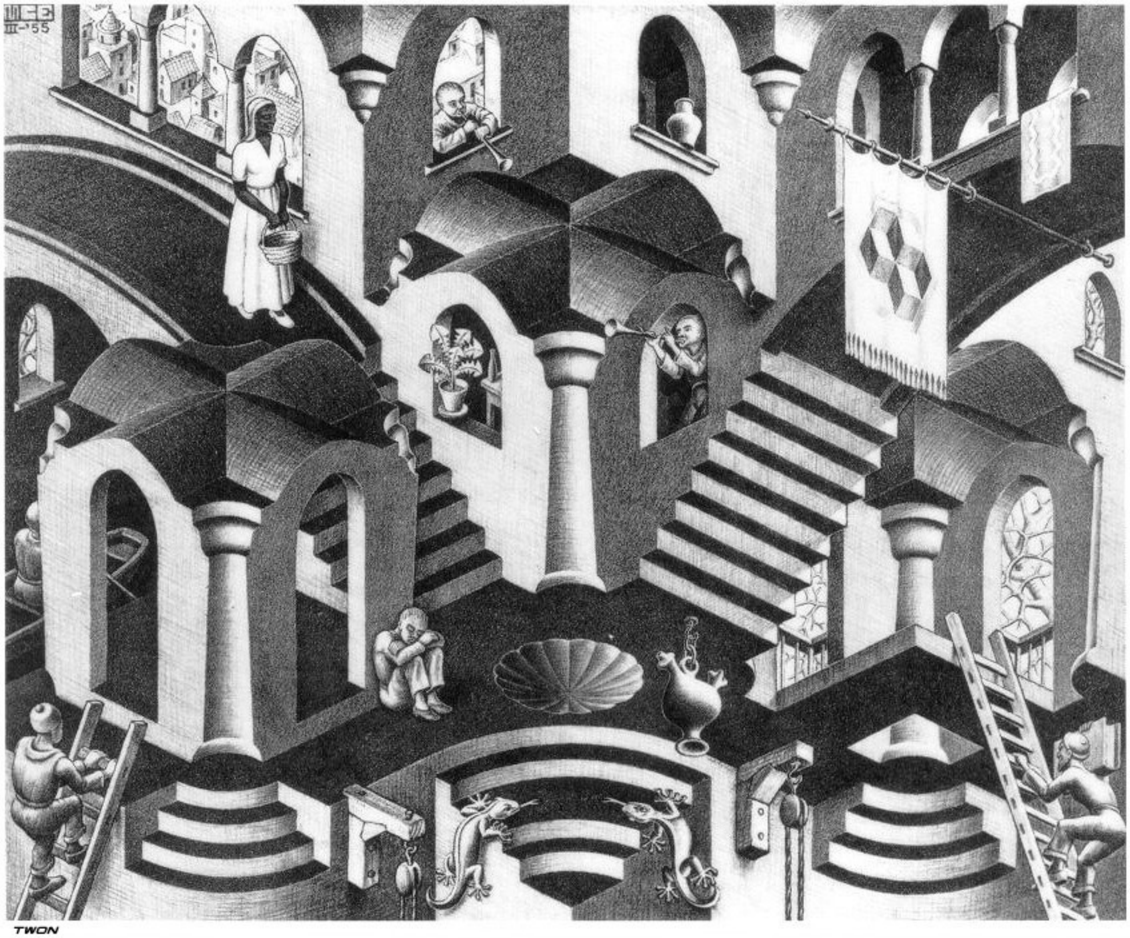


FIGURE 4. The two-edged visualisation in “Convex and concave” by M.C. Escher.

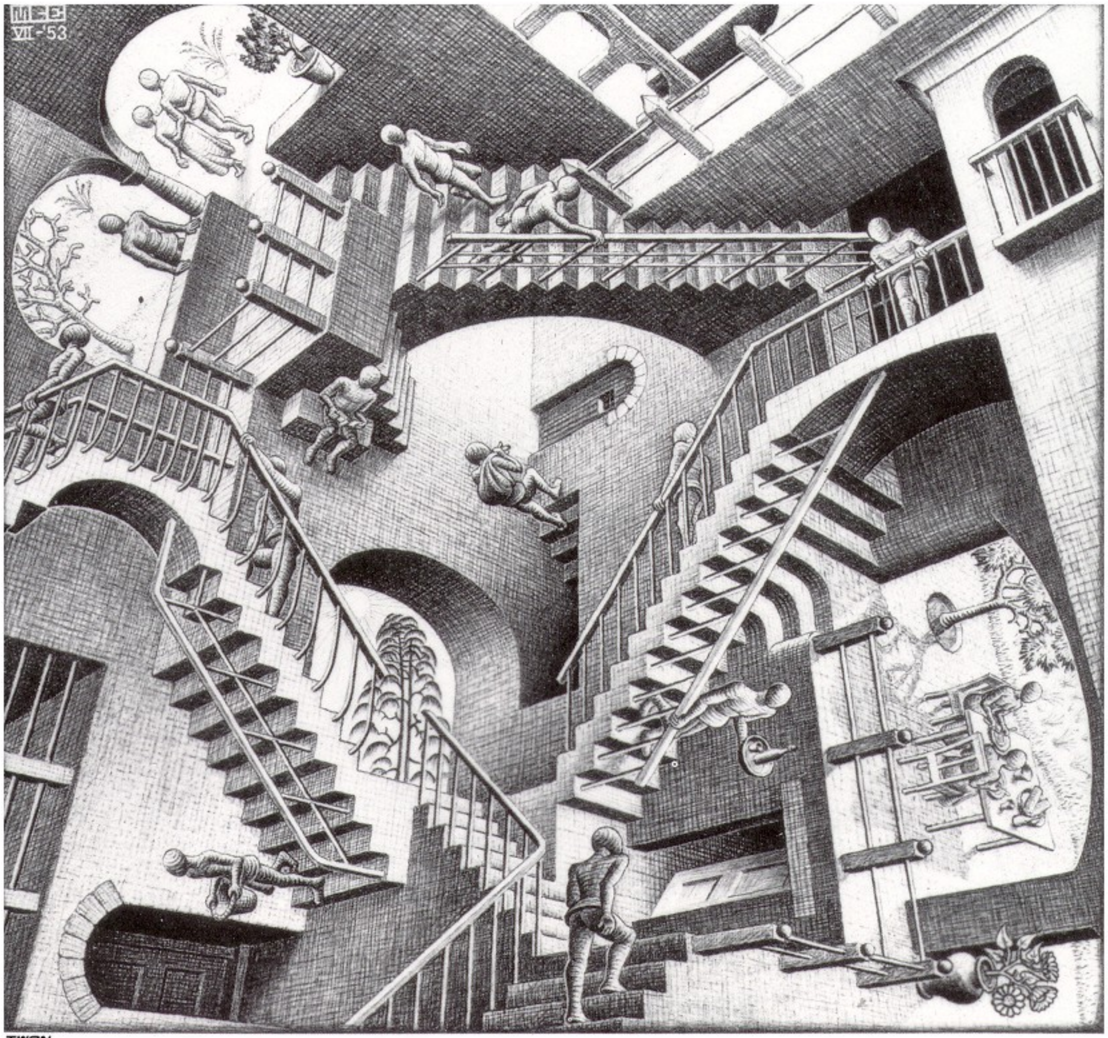


FIGURE 5. The two-edged visualisation in "Relativity" by M.C. Escher.

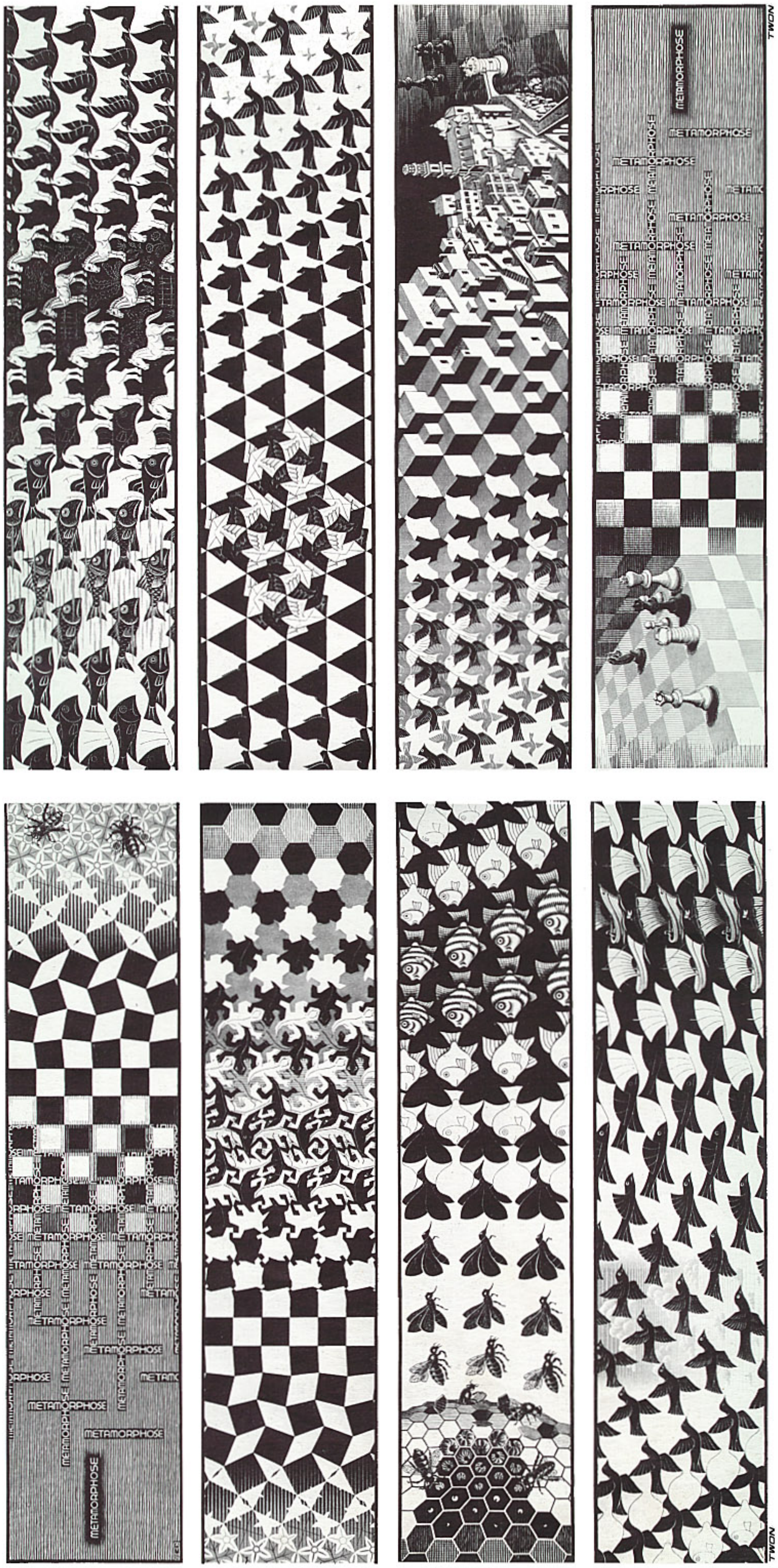


FIGURE 6. The variable dimension visualisation in "Metamorphosis III" by M.C. Escher.

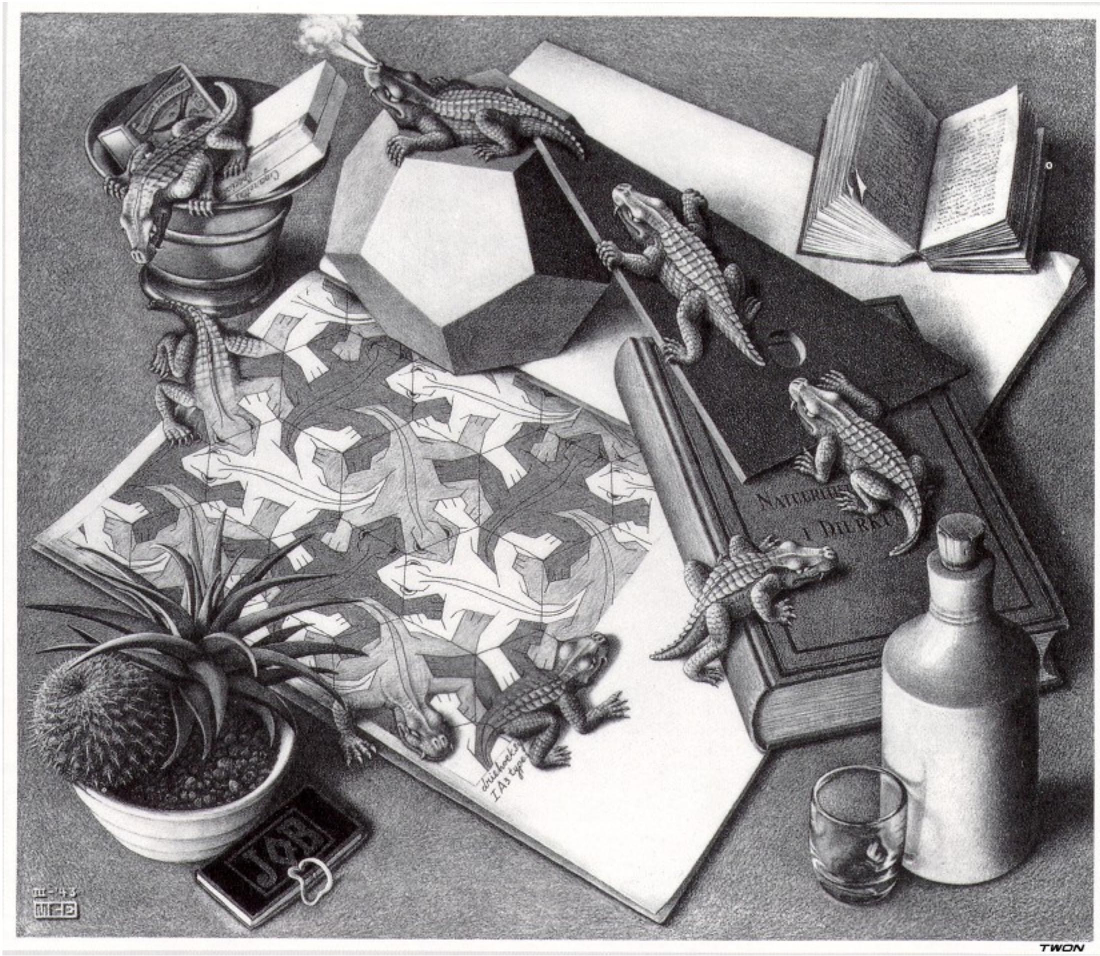


FIGURE 7. The variable dimension visualisation in “Reptiles” by M.C. Escher.

The next step was done by Douglas Dunham in the paper [14], according to which "M.C. Escher was the first person to do so [art in the hyperbolic plane], doing all the needed constructions laboriously by hand. To exhibit the true hyperbolic nature of such art, the pattern must exhibit symmetry and repetition. Thus, it is natural to use a computer to avoid the tedious hand constructions performed by Escher. We show a number of hyperbolic patterns, which are created by combining mathematics, artistic considerations, and computer technology". Saul Schleimer and Henry Segerman constructed a number of sculptures, each based on a geometric design native to the three-dimensional sphere [51]. Using stereographic projection they transferred the design from the three-sphere to ordinary Euclidean space. All of the sculptures were then fabricated by the 3D printing service Shapeways. The authors considered their sculptures as tangible representives of topological and geometric abstractions. Roice Nelson and Henry Segerman in [44] described ways to visualize three-dimensional generalized regular tilings in spherical, Euclidean or hyperbolic spaces.

Jeff Weeks in [59] argues about these spaces as follows.

"Curly leaf lettuce makes an excellent model of the hyperbolic plane. You can make yourself a paper model of the hyperbolic plane by cutting a large number of equilateral triangles from a few sheets of paper and taping the triangles together so that exactly seven triangles (not six) meet at each vertex in the resulting surface. Either way, with lettuce or a paper surface, the hyperbolic plane is roomy in the sense that the area A enclosed by a circle of radius r grows much faster than the expected $A = \pi r^2$. In fact, it quickly approaches an exponential growth rate.... Beyond its many applications in math and physics, hyperbolic space has recently found a new application in data visualization [43]. Its rapidly growing volume provides an excellent environment for displaying large data sets. For example, a large binary tree can't be embedded in flat space without extreme crowding because the number of nodes grows exponentially as a function of the depth, but the tree fits comfortably in hyperbolic space with no crowding at all because the volume of hyperbolic space also grows exponentially as a function of the radius".

"Astrophysicists already use curved-space graphics as part of their research to determine the shape of the real universe, which is still unknown. In particular, curved space visualizations have provided new insights into the possible topologies for a spherical universe.

I hope that in the future these tools will find additional application in computer graphics and games. The hypersphere offers gamers the novelty of curved space along with the convenience of a finite yet boundaryless space, while hyperbolic space provides the roominess for working with exponentially complicated data sets".

Indeed, John Lamping, Ramana Rao, and Peter Pirolli in papers [35] and [36] described a convenient focus + context browser for visualizing and manipulating large hierarchies, placed in the hyperbolic plane. A similar tool to interactively visualize large directed graphs in three-dimensional space was Walrus [56], which used 3D hyperbolic geometry to a display that simultaneously showed local detail and the global context.

On the other hand, according to [47], "The main effort for mathematics visualization, particularly of Non-Euclidean spaces, took place at the Geometry Center[2] from 1994 to 1998. This initiative, under the leadership of William Thurston, resulted in a scientific program to study and disseminate modern geometry using interactive visualization. ... For this purpose, a platform called Geomview [1] was developed. The software was based on OpenGL and supported interactive viewing in Euclidean, spherical, and hyperbolic spaces". Geomview featured a plugin architecture that made possible, among other things, the development of a module for the visualization of manifolds [22].

[2]The National Science and Technology Research Center for Computation and Visualization of Geometric Structures (USA).

Another step are dynamic visualisations. In [23] (see also [20]) are mentioned different projects at the Geometry Center. "One is the video Not Knot [21]. This video, whose purpose is to illustrate some of the basic concepts of knot theory and the theory of 3-manifolds, includes a fly-through scene of hyperbolic 3-space ... During this fly-through one easily notices that apparent size changes more rapidly in hyperbolic space than in Euclidean space. Angles appear to change a we move closer to them. In fact, however, they are no changing – what changes is our perception of them.

Another project ... is a flight simulator for hyperbolic space written by Linus Upson, a Princeton University undergraduate working as a research assistant during the summer of 1991. Patterned after the popular SGI flight simulator, Upson's program allows one to navigate through a scene in hyperbolic space ... The program is excellent for conveying a sense of how angles and distances seem to change with motion. The intuition which one gains from this experience is hard to pinpoint but extremely valuable in understanding hyperbolic geometry". Jeff Weeks' software Curved Spaces [57] is a "flight simulator for multiconnected universes". It simulates movement within a selection of closed three-dimensional manifolds, with $\mathbb{S}^3$, $\mathbb{E}^3$ and $\mathbb{H}^3$ geometries [that is homogeneous and isotropic: spherical, three-dimensional euclidean and hyperbolic geometries]. Each of these is viewed as if we are living inside the space and seeing objects in the space via rays of light that travel long geodesics in the space. That is, light travels along paths of shortest distance" [25].

The character of the computer game HyperRogue [66],[6] by Zeno Rogue wanders about the hyperbolic plane. The vast game space provides a lot of adventures, see [67] and [68] for nice illustrations based, as we understood, on different models of the hyperbolic geometry.

Moreover, even more exotic geometries are implemented in the game, see [67]. The deep mathematical motivation for this is explained in [7]: "Non-isotropic geometries do not behave the same in all directions. Although, they are less famous than the isotropic geometries, they arise in Thurston's famous geometrization conjecture [54]. This conjecture generalizes the Poincaré conjecture, one of the most important conjectures in mathematics, proven by Perelman [49]. Every two-dimensional compact manifold can be given a spherical $\mathbb{S}^2$, Euclidean, or hyperbolic geometry $\mathbb{H}^2$; the Thurston conjecture states that every three-dimensional compact manifold can be similarly decomposed into subsets, each of which admitting one of eight geometries, called the Thurston geometries. The eight geometries include the three isotropic geometries mentioned, two product geometries ($\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, also called $\mathbb{S}^2 \times \mathbb{E}$ and $\mathbb{H}^2 \times \mathbb{E}$), and three other geometries: **Solv**, **Nil** (twisted $\mathbb{E}^2 \times \mathbb{R}$), and twisted $\mathbb{H}^2 \times \mathbb{R}$ (also called the universal cover of $SL(2, \mathbb{R})$). The interest in **Solv** and **Nil** ranges from low-dimensional topologists, geometric group theorists, as those geometries exhibit growth patterns typical to solvable and nilpotent groups [38], to physicists [18], and cosmologists, as possible geometries of our Universe [58][3]. Note that not all three-dimensional geometries are Thurston geometries. There are also non-isotropic geometries for which there are no compact manifolds which admit these geometries....

Other than the scientific purposes, the visualization of non-isotropic geometries has potential applications in video games or art. Many popular (mostly independent) video games experiment with spaces that work differently from our Euclidean world. This includes spaces with weird topology (Portal, Antichamber, Manifold Garden), interactions between 2D and 3D (Perspective, Fez, Monument Valley), non-Euclidean geometry (HyperRogue), extra dimensions (Miegakure). Similar experimentation also happens in art. Such games and art are interesting not only for mathematicians and physicists wanting to understand these spaces intuitively, but also for casual players curious to challenge their perception of the world. Non-isotropic geometries are especially relevant here because of their easily observable weirdness. **Nil**, a reminiscent of Penrose's staircases and M.C. Escher's artworks, should be promising for game design".

[3]Moreover, according to [53], "Although the observed universe appears to be geometrically flat, it could have one of 18 global topologies. A constant-time slice of the spacetime manifold could be a torus, Mobius strip, Klein bottle, or others".

Pierre Berger provided two-dimensional illustrations for all Thurston geometries except twisted $\mathbb{H}^2 \times \mathbb{R}$ in [3]. "However, those visualizations are static images rather than real-time rendered, which makes them difficult to interpret" [7], "it is not clear to me what structures are shown in the picture" [67]. Jeff Weeks presented his visualizations of the isotropic and product spaces and planned to work on the remaining geometries, see [60]. The paper [42] by Emil Molnár and David Papp is devoted to modelling **Nil**-geometry in Euclidean space with software presentation.

"In comparison to two-dimensional non-Euclidean geometries, non-isotropic three-dimensional geometries, **Solv**, **Nil**, and twisted $\mathbb{H}^2 \times \mathbb{R}$, are even more demanding to comprehend. Weeks [58] describes the **Solv** geometry as *"This is the real weirdo. [...] I don't know any good intrinsic way to understand it."*. Therefore, efficient visualization becomes a fundamental tool for gaining intuition about those geometries....

As a result, while there are implementations of real-time first-person view for Euclidean, spherical, hyperbolic spaces [24, 59], and for product spaces [60], real-time visualizations of geometries like **Solv**, **Nil** or twisted $\mathbb{H}^2 \times \mathbb{R}$ were absent until recently." [7]

As far as we know, HyperRogue by ZenoRogue and based by their early work SolvView by MagmaMcFry [40] are the first interactive geodesic visualizations for **Solv** and **Nil** geometries, SolvView also deals with the twisted hyperbolic geometry. The project was continued in [8].

The next step is to simulate the curved spaces in virtual realities and there are several related projects. According to [47], "Researchers at the Geometry Center, already at that time realized the potential of Virtual Reality for providing insights into the world of curved spaces. They created simple VR installations to allow the user, not only to have a glimpse at the visual landscape inside a 3-manifold, but also to experience the sensation of being immersed in such an environment. Two of their projects are Mathenautics [16] and Alice [17]". Also JReality [4], a Java based 3D scene graph package designed for mathematical visualization at TU-Berlin, can be used for creating immersive views of 3-manifold.

"It is worth mentioning that being inside a curved space is far more intuitive than seeing it on a display. For example, deformations are created when we rotate our head inside such spaces – this is due to their non-isotropic nature. Also, walking in the space using the Euclidean coordinates shows how different it is from the classic geometries. The Sol space, in particular, is surprising" [47]. On the other hand, "Initial user testing suggests that people can navigate in hyperbolic virtual environments without major disorientation and may navigate branching structures more intuitively than in Euclidean space" [30].

Sometimes we need to solve specific problems to simulate curved virtual reality. For instance, according to [62], "Virtual-reality simulations of curved space are most effective and most fun when presented as a game (for example, curved-space billiards), so the user not only has something to *see* in the curved space, but also has something fun to *do* there. However, such simulations encounter a geometrical problem: they must track the player's hands as well as her head, and in curved space the effects of holonomy would quickly lead to violations of *body coherence*. That is, what the player sees with her eyes would disagree with what she feels with her hands. This article presents a solution to the body coherence problem, as well as several other questions that arise in interactive VR simulations in curved space". Also Jeff Weeks improved the exploration of curved spaces from [24], by presenting a framework that allows the development of games [61, 63].

Thurston's conjecture motivates the project "to develop accurate, real time, intrinsic, and mathematically useful illustrations of homogeneous (pseudo)-Riemannian spaces" [11], [12], see also [28], [24], [25], [26], and [27]. Its realisation is quite successful, in [12] is reported: "We developed virtual reality software whose aim is to simulate these eight geometries. We populate each of these spaces X with various objects (spheres, planes, cylinders, lights, lattices, etc.) and compute what an observer would see if light follows the geodesics of X. Using a virtual reality headset, the user can walk in these spaces and experience their surprising properties". For an other similar project, involving the ray tracing technology, see Ray

VR [50], and works [39], [45], [46], and [47] by Djalma Lucio, Tiago Novello, Vinícius Silva, and Luiz Velho. In the latter paper the following perspectives to extend the ray tracing approach are discussed. "This paper deals with mathematical visualization, however, in this same line there is the interest of visualizing physical concepts. The four-dimensional *space-time* is one of these. Weiskopf [64] used ray tracing to explore space-time by simulating travels faster than the light speed. Gröller [19] visualized relativistic effects, the geometric behavior of nonlinear dynamical systems, and the movement of charged particles in a force field (e.g., electron movement). Additionally, the website [10] presents some interesting relativistic images. However, those visualizations are not in real-time. Approaching this problem using our framework could be an interesting challenge. Another one is the visualization of spaces related to *string theory* [5], especially the concepts of *D-branes* [34]".

Moreover, some thinkers, for instance, Charles Howard Hinton (see [9] and [15]) (1853–1907) and Piotr Uspenskiy (1878–1947) (see, for instance [55]) considered possibility of four-dimensional visualization. Note that a few years ago was discovered that neural networks of our brains indeed have some potential for it, because they can create structures in up to 11 dimensions, see [13]. Hinton argued that we can develop a skill for four-dimensional visualization by freeing our imagination of objects of "elements of the self", related to our vision and location, in order to imagine the things as they are, for instance, to see not only their surfaces but also inner points. This aim can be achieved by special imagination exercises. "Hinton later introduced a system of coloured cubes by the study of which, he claimed, it was possible to learn to visualise four-dimensional space [32]. Rumours subsequently arose that these cubes had driven more than one hopeful person insane" [9]. Luckily, the second author is a mad scientist, so this scenario fits to him. But now he is finishing a habilitation thesis under the guidance of the first author, and the possible problem would decrease important statistical data for his institute. So we had to postpone the project for a few of years.

Our contribution. Meantime, Bogdan Okhrimenko, under the supervision of the first author and Iryna Novozhylova, wrote a Java program to help to produce four-dimensional visualisations of 1-skeletons of four-dimensional regular convex polyhedra (see Fig. 9) [48]. The program provides projections of the rotating polyhedron on two orthogonal planes in the four-dimensional space. If to each eye is given a separate image (see Fig. 10) then they potentially can be joined to produce a four-dimensional image, similarly to the well-known practice to see three-dimensional images.

In order to explain the approach from [48], we start from the three-dimensional case. From the times of Euclid (III cent. BC) it is known that there are exactly five types of three-dimensional regular convex polyhedra [2], known as Platonic solids:

Polyhedron	Face	Vertices	Edges	Faces
Tetrahedron	triangle	4	6	4
Cube	square	8	12	6
Octahedron	triangle	6	12	8
Dodecahedron	pentagon	20	30	12
Icosahedron	triangle	12	30	20

Platonic solids can be realized in three-dimensional Euclidean space $\mathbb{R}^3$ by centering them at the origin and providing the coordinates of their vertices and edges between them, for instance, as follows:

- for a *tetrahedron* we place its four vertices at the points $(1, 1, 1)$, $(-1, -1, 1)$, $(-1, 1, -1)$, and $(1, -1, -1)$ and connect any two vertices by an edge iff the distance between them is $2\sqrt{2}$;
- for a *cube* we place its eight vertices at the points $(\pm 1, \pm 1, \pm 1)$ and connect any two vertices by an edge iff the distance between them is 2;

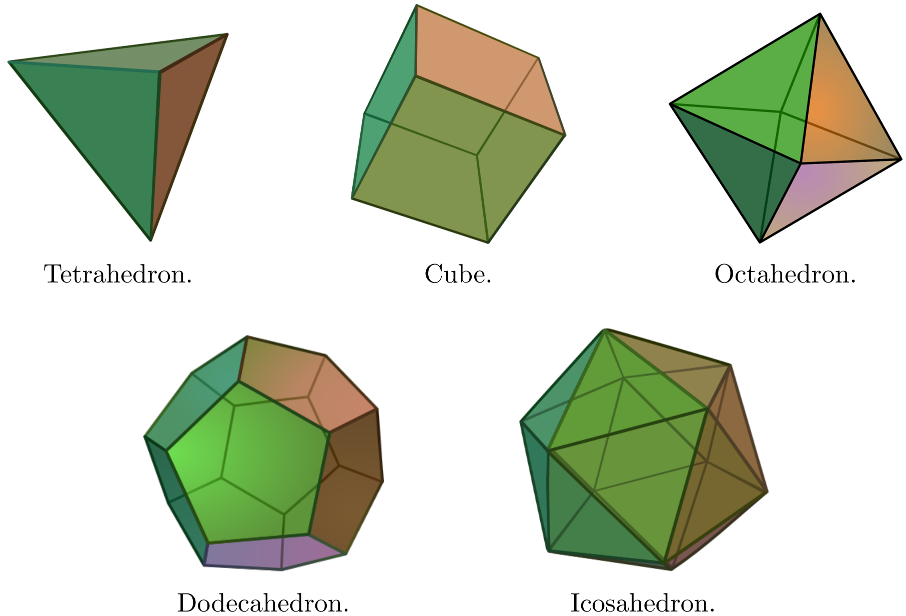


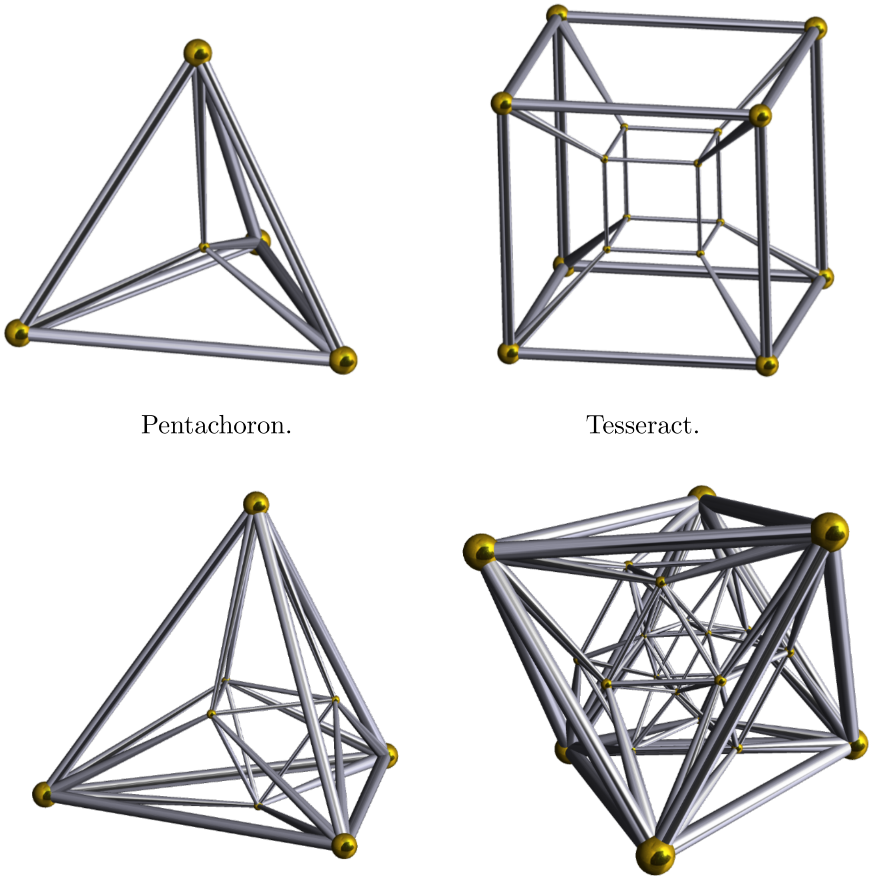
FIGURE 8. Platonic solids [65].

- for an *octahedron* we place its six vertices at the points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, and $(0, 0, \pm 1)$ and connect any two vertices by an edge iff the distance between them is $\sqrt{2}$.

Going now into the four-dimensional space, recall that by Schläfli theorem [2, 12.6.7], there are exactly six types of four-dimensional regular convex polyhedra [4]

Polyhedron	Cell	Vertices	Edges	Faces	Cells
Pentachoron	tetrahedron	5	10	10	5
Tesseract	cube	16	32	24	8
Orthoplex	tetrahedron	8	24	32	16
Octaplex	octahedron	24	96	96	24
Dodecaplex	dodecahedron	600	1200	720	120
Tetraplex	tetrahedron	120	720	1200	600

[4]Note that more dimensions do not provide more types, because the theorem implies that for any $n \geq 5$ there are only three types of n-dimensional regular convex polyhedra, namely, the n-dimensional simplex, the n-dimensional cube, and its dual, the n-dimensional cocube. These regular polyhedra are n-dimensional counterparts of the tetrahedron, the cube, and the octahedron, respectively.



Pentachoron. Tesseract.

Orthoplex. Octaplex.

FIGURE 9. 1-skeletons of the four-dimensional regular polyhedra [48].

Note that a pentachoron is a four-dimensional simplex, tesseract is also called a four-dimensional cube or a hypercube, and an orthoplex is a dual polyhedron to a tesseract.

Similarly to Platonic solids, the four-dimensional regular convex polyhedra can be realized in four-dimensional Euclidean space $\mathbb{R}^4$ by centering them at the origin and providing the coordinates of their vertices and edges between them, for instance, as follows:

- for a *pentachoron* we place its five vertices at the points $(-1,-1,-1,1/\sqrt{5})$,$(-1,1,1,1/\sqrt{5})$, $(1,-1,1,1/\sqrt{5})$, $(1,1,-1,1/\sqrt{5})$, and $(0,0,0,-4/\sqrt{5})$ and connect any two vertices by an edge iff the distance between them is $2\sqrt{2}$;
- for a *tesseract* we place its sixteen vertices at the points $(\pm1,\pm1,\pm1,\pm1)$ and connect any two vertices by an edge iff the distance between them is 2;
- for an *orthoplex* we place its eight vertices at the points $(\pm1,0,0,0)$, $(0,\pm1,0,0)$, $(0,0,\pm1,0)$ and $(0,0,0,\pm1)$ and connect any two vertices by an edge iff the distance between them is $\sqrt{2}$;
- for an *octaplex* we place its twenty four vertices at the points $(\pm1,\pm1,0,0)$, $(\pm1,0,\pm1,0)$, $(\pm1,0,0,\pm1)$, $(0,\pm1,\pm1,0)$, $(0,\pm1,0,\pm1)$, and $(0,0,\pm1,\pm1)$ and connect any two vertices by an edge iff the distance between them is $\sqrt{2}$.

The program provides projections of the 1-skeleton of a given rotating polyhedron on two orthogonal planes in the four-dimensional space. The colors of vertices allow to track their trajectories. Note that for the centrally symmetric polyhedra (tesseract, orthopex, and octaplex), the antipodal vertices have the same color. We set the size of the vertex projection depending on the distance from the vertex to the plane to project.

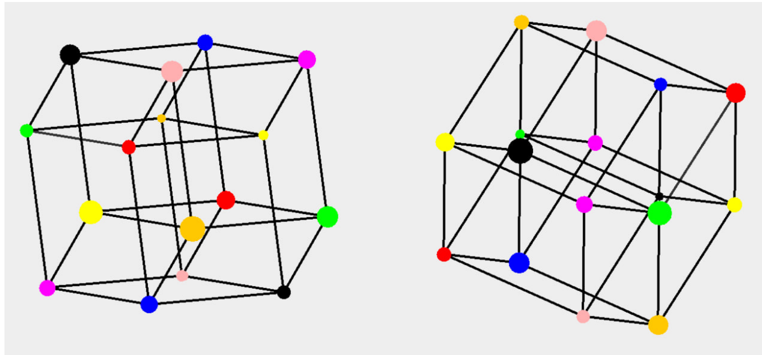


FIGURE 10. The projections of the rotated tesseract to the planes Oxy and Ozw [48].

The videos (in mkv format) of the rotating polyhedra can be downloaded via the following links:

Polyhedron	Size	URL
Pentachoron	18.2 MB	mega.nz/file/RsBmRThB#0T43DF5W820zlR4D1Qgvg2Ys2fLaWjWGcgqd8CZ0RAY
Tesseract	21.7 MB	mega.nz/file/ppJ03LTD#WemGYXiFRHG_Zw3AjVt0olx-kc1bZkrIYcDdxScUH_8
Orthoplex	14.2 MB	mega.nz/file/F0RH0BCY#WCrG0c7-gWxUppfJ0Q3rnWxWuYZuoQYwIcFMjzKXloc
Octaplex	19.6 MB	mega.nz/file/lkJGya6S#5yvk10mycQJBSULGkfmBK6H6n07r0GKKP1UCjR7xUfc

Future work. Note that from philosophical point of view, the possibility of nonstandard visualization refutes the Kantian teaching that three-dimensional Euclidean space is a priori format of our experience. On the other hand there is the Buddhist teaching on shunyata, which is ”an infinitely open space or background that allows for anything to appear, change, disappear, or reappear” [41]. Taking into account both this issue and visualization subtleness, we hope to involve to the project more experienced contemplators from European branch [5] of Mind & Life Institute, founded by “Tenzin Gyatso, the 14th Dalai Lama–the spiritual leader of the Tibetan people and a global advocate for compassion; Francisco Varela, a scientist and philosopher; and Adam Engle, a lawyer and entrepreneur”, which “bring science and contemplative wisdom together to better understand the mind and create positive change in the world”. The joint project can be especially interesting because we think that, according to Tibetan Buddhism, nonstandard visualisation experience can help a person to attain liberation, see the appendix for details.

Acknowledgments. The authors thank to Sara Di Bartolomeo for informing them on alt.VIS workshop, to Volodymyr Pelykh for pointing them the papers [13] and [53], and to Ivan Hetman for pointing them the game HyperRogue.

Appendix: Nonstandard visions in Tibetan Buddhism

Dzongsar Jamyang Khyentse in [33] says that traditionally, Tibetans rely on the instructions that appear in Karma Lingpa’s “Great Liberation through Hearing in the Bardo”, the “Bardo Tödrol Chenmo”, more known at the West as“The Tibetan Book of the Dead”, see also [37] and [52]. Unfortunately, after the paper was accepted, we found that the author of [52], Sogyal Rinpoche, is accused of immorality (see, for instance, [29]). So, although there still is a hope that the quotations from [52] below conform to Tibetan Buddhism, please be careful reading them.

In particular, a dying person at some phase is instructed as follows, see [33, The Painful Bardo of Dying: Dharmakaya]:

> “The infinite rainbow-like colours and shapes that now surround you
> Are unlike anything you have ever seen before.

[5] See mindandlife-europe.org.

The blueness of the blue,
The greenness of the green,
The redness of the red
Are unimaginably intense and alive.
Because you are no longer limited by the filter of your eyes
You are able to perceive all the unnamed colours
That were invisible to you while you were alive.
...
You can see some familiar shapes,
Like squares, triangles and semi-circles,
But most are completely unfamiliar to you;
You never imagined that such shapes exist.
...
Everything feels intense and raw
Because you no longer perceive
Using the filters of your body's sense organs,
Or your imagination.
There is nothing between you and the object you are experiencing.
...
Do not be afraid of the colours and shapes,
Or of how intensely you perceive them.
They are nothing more than the expression of your mind ...
Nothing you see and experience is 'out there',
It is all the radiant display of mind.
...
Do not be afraid.
There is no need to panic.

You will now faint.

O, Son or Daughter of Noble Family,...
[name of dying person],
This is the Buddha!
...
Do not be afraid!
Do not contrive!

This is the Buddha;
This is the real you!
You are not [name of dying person].
You are Buddha,
Face it!
Dwell in your true nature!

You are Buddha,
Do not shy away from your buddha nature!

This is it!

Do not try to run away from this state!
Relax and dwell right here".

Sogyal Rinpoche in [52, Bardos and Other Realities] explains:

"The reason the moment of death is so potent with opportunity is because it is then that the fundamental nature of mind, the Ground Luminosity or Clear Light, will naturally manifest, and in a vast and splendid way. If at this crucial moment we can recognize the Ground Luminosity, the teachings tell us, we will attain liberation. This is not, however, possible unless you have become acquainted and really familiar with the nature of mind in your lifetime through spiritual practice.

And this is why, rather surprisingly, it is said in our tradition that a person who is liberated at the moment of death is considered to be liberated in *this* lifetime, and *not* in one of the bardo states after death; for it is within this lifetime that the essential recognition of the Clear Light has taken place and been established. This is a crucial point to understand".

Then Sogyal Rinpoche argues that a person has to be prepared to the moment of death during the lifetime, by acquiring the similar experience to be acquainted with it. Dzongsar Jamyang Khyentse explains: "Most of us prefer to stick to what we are used to. Although the emotions we habitually experience can be agonizingly painful, they are also comfortingly familiar. More often than not, we would rather experience the pain we know than nothing at all – mind is so masochistic. This is why the 'referencelessness' we experience once our bodies are dead is so unbearable... we feel far more comfortable with the less intimidating, not-too-bright and not-too-extraordinary colours, figures and shapes that we now see, and why we long to cosy up to them". Note that the nonstandard visualisation experience makes exotic shapes familiar and so more attractive.

Sogyal Rinpoche notes that at the moment of death "is a very special state of luminosity or Clear Light called, as I have said, the "bardo of dharmata." This is an experience that occurs for everyone, but there are very few who can even notice it, let alone experience it completely, as it can only be recognized by a trained practitioner. This bardo of dharmata corresponds to the period after falling asleep and before dreams begin". The second author confirms that he indeed sometimes sees some surrealistic and transforming shapes while he is falling asleep.

The above suggests that the nonstandard visualisation experience can help a person to attain liberation.

References

[1] N. Amenta, S. Levy, T. Munzner, M. Phillips, *Geomview: a system for geometric visualization*, In Proceedings of the eleventh annual symposium on Computational geometry SCG'95, 412–413, ACM 1995.

[2] M. Berger, *Geometry I*, Springer, 2009.

[3] Pierre Berger, *Espaces imaginaires*, 2015, `espaces-imaginaires.fr/works/ExpoEspacesImaginaires2.html`.

[4] P. Brinkmann, C. Gunn, T. Hoffmann, U. Pinkall, S. Weismann, *JReality: a Java library for real-time interactive 3D graphics and audio*, In ACM Multimedia'09, 2009, 927–928.

[5] V. Burkin, H. Hagen, S. Klimenko, I. Nikitin, V. Semenov, O. Tarlapan, *Visualization in string theory*, Computers & Graphics **24** (2000), 23–30.

[6] Dorota Celińska, Marek Čtrnáct, Eryk Kopczyński, *HyperRogue: Playing with Hyperbolic Geometry*, In Proceedings of Bridges 2017: Mathematics, Art, Music, Architecture, Education, Culture, eds. David Swart, Carl H. Séquin, Kristóf Fenyvesi, 9–16, Phoenix, AZ: Tesselations Publishing, `archive.bridgesmathart.org/2017/bridges2017-9.pdf`.

[7] Dorota Celińska-Kopczyńska, Eryk Kopczyński, *Real-Time Visualization in Non-Isotropic Geometries*, arXiv, `arxiv.org/abs/2002.09533`.

[8] Dorota Celińska-Kopczyńska, Eryk Kopczyński, *Real-Time Visualization in Anisotropic Geometries*, Experimental Mathematics **31**:4 (2022) 1177–1196, doi:10.1080/10586458.2022.2050324.

[9] *Charles Howard Hinton*, `geni.com/people/Charles-Howard-Hinton/6000000017535087228`.

[10] M. Cicconet, *Visualização relativística usando ray tracing*, 2007, `lvelho.impa.br/i3d07/demos/cicconet`.

[11] Remi Coulon, Elisabetta A. Matsumoto, Henry Segerman, Steve Trettel, *Non-Euclidean Virtual Reality III: Nil*, arXiv, `arxiv.org/abs/2002.0051`.

[12] Remi Coulon, Elisabetta A. Matsumoto, Henry Segerman, Steve Trettel, *Non-Euclidean Virtual Reality IV: Sol*, arXiv, `arxiv.org/abs/2002.00369`.

[13] Signe Dean, *Scientists Find Evidence The Human Brain Can Create Structures in Up to* 11 *Dimensions*, ScienceAlert, 31 August 2019, `sciencealert.com/scientists-find-evidence-the-human-brain-can-create-structures-in-up-to-11-dimensions`.

[14] Douglas Dunham, *Artistic Patterns in Hyperbolic Geometry*, In Procceding of Bridges 1999: Mathematical Connections in Art, Music, and Science, eds. Reza Sarhangi, 239–250, Winfield, `archive.bridgesmathart.org/1999/bridges1999-239.pdf`.

[15] Eric Eldred, *Charles H. Hinton*, `ibiblio.org/eldritch/chh/hinton.html`.

[16] T.A. DeFanti, G.K. Francis, C. Gunn, R. Hudson, D.J. Sandin, *Mathenautics: using VR to visit 3-D manifolds*, In Proceedings of the 1995 symposium on Interactive 3D graphics I3D'95, 167–170, New York: ACM 1995.

[17] G.K. Francis, C.M.A. Goudeseune, H.J. Kaczmarski, B.J. Schaeffer, J.M. Sullivan, *Alice on the eightfold way: Exploring curved spaces in an enclosed virtual reality theatre*, In Visualization and Mathematics III 2003, 305–315.

[18] J. Gegenberg, S. Vaidya, J.F. Vázquez-Poritz, *Thurston geometries from eleven dimensions*, Classical and Quantum Gravity **19**:23 (Nov 2002 L199–L204, doi:10.1088/0264-9381/19/23/102.

[19] E. Gröller, *Nonlinear ray tracing: Visualizing strange worlds*, The Visual Computer **11**:(5) (1995) 263–274.

[20] Charlie Gunn, *Visualizing hyperbolic space*, In Computer Graphics and Mathematics, eds. B. Falcidieno et al., 299–311, Springer-Verlag 1991.

[21] Charlie Gunn et. al., *Not Knot*, videotape, Jones and Bartlett. (Copies of this video may be ordered by contacting Jones and Bartlett Publishers, Inc, 20 Park Plaza, Suite 1435, Boston, MA 02116-9792).

[22] C. Gunn, *Discrete groups and visualization of three-dimensional manifolds*, In Proceedings of the 20th annual conference on Computer graphics and interactive techniques, SIGGRAPH'93, 255–262, New York: ACM 1993, doi:10.1145/166117.166150.

[23] Charlie Gunn, Mark Phillips, *Visualizing hyperbolic space: Unusual Uses of* 4×4 *Matrices*, Proceedings of the 1992 Symposium on Interactive 3D Graphics, 209–214, `researchgate.net/profile/Charles_Gunn/publication/220792023_Visualizing_Hyperbolic_Space_Unusual_Uses_of_4x4_Matrices/links/58a7605292851cf0e3b9730a/Visualizing-Hyperbolic-Space-Unusual-Uses-of-4x4-Matrices.pdf`, doi:10.1145/147156.147206.

[24] Vi Hart, Andrea Hawksley, Elisabetta A. Matsumoto, Henry Segerman, *Non-euclidean Virtual Reality I: Explorations of* $\mathbb{H}^3$, In Proceedings of Bridges 2017: Mathematics, Art, Music, Architecture, Education, Culture, eds. David Swart, Carl H. Séquin, Kristóf Fenyvesi, 33–40, Phoenix, AZ: Tesselations Publishing, `archive.bridgesmathart.org/2017/bridges2017-33.pdf`.

[25] Vi Hart, Andrea Hawksley, Elisabetta A. Matsumoto, Henry Segerman, *Non-euclidean Virtual Reality II: Explorations of* $\mathbb{H}^2 \times \mathbb{E}$, In Proceedings of Bridges 2017: Mathematics, Art, Music, Architecture, Education, Culture, eds. David Swart, Carl H. Séquin, Kristóf Fenyvesi, 41–40, Phoenix, AZ: Tesselations Publishing, `archive.bridgesmathart.org/2017/bridges2017-41.pdf`.

[26] Vi Hart, Andrea Hawksley, Elisabetta A. Matsumoto, Henry Segerman, *Non-euclidean virtual reality*, The talk at Bridges 2017, `youtube.com/watch?v=72SGtw5ca1c`.

[27] Vi Hart, Andrea Hawksley, Elisabetta A. Matsumoto, Henry Segerman, *Non-euclidean virtual reality*, 2017, `youtube.com/watch?v=ztsi0CLxmjw`

[28] Vi Hart, Andrea Hawksley, Henry Segerman, Marc ten Bosch, *Hypernom: Mapping VR headset orientation to* $\mathbb{S}^3$, In Proceedings of Bridges 2015: Mathematics, Art, Music, Architecture, Culture, 387–390, Tesselations Publishing, `archive.bridgesmathart.org/2015/bridges2015-387.pdf`.

[29] Oliver Harvey, *The bad Buddha: Dark side of celeb guru Sogyal Rinpoche who 'sexually abused' the beautiful young women dubbed his 'Dakinis'*, The Sun (UK edition), 22 Sep 2018, `thesun.co.uk/news/7319165/celeb-guru-sogyal-rinpoche-sexually-abused-women-dubbed-dakinis`.

[30] Nico Hawthorne, Ocean Hurd, Sri Kurniawan, Vincent Adelizzi Pisani, *Navigation by Walking In Hyperbolic Space Using Virtual Reality*, In Extended Abstracts of the Annual Symposium on Computer-Human Interaction in Play Companion Extended Abstracts, CHI PLAY'19, October 22–25, 2019, Barcelona, Spain, doi:10.1145/3341215.3356287.

[31] Linda Dalrymple Henderson, *The Fourth Dimension and Non-Euclidean Geometry in Modern Art*, Rev. ed., Princeton, NJ: Princeton University Press 2012.

[32] Charles Howard Hinton, *Casting out the Self*, 1904.

[33] Dzongsar Jamyang Khyentse, *Living is Dying: How to Prepare for Dying, Death and Beyond*, `siddharthasintent.org/publications/living-is-dying`.

[34] C.V. Johnson, *D-branes*, Cambridge university press 2002.

[35] John Lamping, Peter Pirolli, Ramana Rao, *A Focus+Context Technique Based on Hyperbolic Geometry for Visualizing Large Hierarchies*, In Proceedings of the SIGCHI Conference on Human Factors in Computing Systems, CHI '95, 401–408, New York, NY:ACM Press/Addison-Wesley Publishing Co. 1995, doi:10.1145/223904.223956.

[36] John Lamping, Ramana Rao, *The Hyperbolic Browser: A Focus+Context Technique for Visualizing Large Hierarchies*, Journal of Visual Languages & Computing (1996) **7**:1 33–55, doi:10.1006/jvlc.1996.0003.

[37] Karma Lingpa, *The Tibetan Book of the Dead*, London 1927 (Russian translation, SPb.: Chernyshev publ. 1992).

[38] C. Löh, *Geometric Group Theory: An Introduction*, Universitext. Springer International Publishing, 2017.

[39] Djalma Lucio, Tiago Novello, Vinícius Silva, Luiz Velho, *Visualization of non-euclidean spaces using ray tracing*, Technical Report TR-09-2019; VISGRAF Lab - IMPA 2019, `visgraf.impa.br/Data/RefBib/PS_PDF/tr-09-19/tr-09-2019.pdf`.

[40] MagmaMcFry, *SolvView*, `github.com/MagmaMcFry/SolvView`.

[41] Yongey Mingyur Rinpoche, Eric Swanson, *Joyful Wisdom. Embracing Change And Finding Freedom*, New York: Harmony Books 2009?.

[42] Emil Molnár, David Papp, *Visualization of Nil-geometry: Modelling Nil-geometry in Euclidean Space with Software Presentation*, 2003, doi:10.25596/jalc-2014-201.

[43] T. Munzner, *H3: Laying Out Large Directed Graphs in 3D Hyperbolic Space*, Proc. 1997 IEEE Symp. Information Visualization, 2-10, Los Alamitos, Calif.: IEEE CS Press 1997, `graphics.stanford.edu/papers/h3`.

[44] Roice Nelson, Henry Segerman, *Visualizing hyperbolic honeycombs*, Journal of Mathematics and the Arts **11**:1 (2017) 4–39, `arxiv.org/abs/1511.02851`.

[45] Tiago Novello, Vinícius Silva, Luiz Velho, Immersive Visualization of the Classical Non-Euclidean Spaces using Real-Time Ray Tracing in VR, Proceedings of the 46th Graphics Interface, 2020, `visgraf.impa.br/Data/RefBib/PS_PDF/gi2020/gi2020.pdf`

[46] Tiago Novello, Vinícius Silva, Luiz Velho, *Visualization of Nil, SL2 and Sol*, 2019, `visgraf.impa.br/ray-vr/?page_id=252`

[47] Tiago Novello, Vinícius Silva, Luiz Velho, *Visualization of Nil, Sol, and* $\widetilde{SL_2}(\mathbb{R})$ *Geometries*, Computers & Graphics (2020), `visgraf.impa.br/Data/RefBib/PS_PDF/cag2020a/Visualization_of_Nil_Sol_SL2_CAG.pdf`.

[48] Bogdan Okhrimenko, *To see the fourth dimension*, a competition paper, Drogobych 2023 (in Ukrainian).

[49] G. Perelman, *The Entropy formula for the Ricci flow and its geometric applications*, arXiv math/0211159, 2006.

[50] *Ray VR*, `visgraf.impa.br/ray-vr`.

[51] Saul Schleimer, Henry Segerman, *Sculptures in* S^3, In Proceedings of Bridges 2012: Mathematics, Music, Art, Architecture, Culture, eds. Robert Bosch, Douglas McKenna, Reza Sarhangi, 103–110, Tessellations Publishing, `archive.bridgesmathart.org/2012/bridges2012-103.pdf`

[52] Sogyal Rinpoche, *The Tibetan Book of Living and Dying: Revised and Updated*, London, Rider/Penguin Random House 2017.

[53] P.M. Sutter, Tsunefumi Tanaka, *Vacuum Energy Density for Massless Scalar Fields in Flat Homogeneous Spacetime Manifolds with Nontrivial Topology*, arXiv, `arxiv.org/abs/gr-qc/0610051`.

[54] W.P. Thurston, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bulletin (New Series) of the American Mathematical Society **6**:3 (May 1982) 357–381.

[55] P.D. Uspenskiy, *Tertium Organum. The key to the enigmas of the world*, Sanct-Peterburg: Andreev and sons 1992 (reprinted from 1911?) (in Russian).

[56] *Walrus - Graph Visualization Tool*, `caida.org/catalog/software/walrus`.

[57] Jeff Weeks, *Curved Spaces. A flight simulator for multiconnected universes*, `geometrygames.org/CurvedSpaces`.

[58] J. Weeks, *The Shape of Space*, Chapman & Hall/CRC Pure and Applied Mathematics. Taylor & Francis, 2001; Marcel Dekker 2002, `books.google.com.br/books?id=Lurp6nB4LtQC`

[59] Jeff Weeks, *Real-time rendering in curved spaces*, IEEE Computer Graphics and Applications, **22**:6 (2002) 90–99, doi:10.1109/MCG.2002.1046633.

[60] Jeff Weeks, *Real-time animation in hyperbolic, spherical, and product geometries*, In Non-Euclidean Geometries. Mathematics and Its Applications, ed. M.E.A. Prkopa, **581**, 287–305, Boston, MA: Springer Jan 2006, doi:10.1007/0-387-29555-0_15.

[61] Jeff Weeks, *Non-Euclidean Billiards in VR*, In Proceedings of Bridges 2020: Mathematics, Music, Art, Architecture, Culture, 1–8, `archive.bridgesmathart.org/2020/bridges2020-1.pdf`.

[62] Jeff Weeks, *Body coherence in curved-space virtual reality games*, arXiv, `arxiv.org/abs/2011.00510`.

[63] Jeff Weeks, *Virtual reality simulations of curved spaces*, in preparation (2020).

[64] D. Weiskopf, *Visualization of four-dimensional spacetimes*, University of Tübingen 2001.

[65] Wikipedia, *Platonic solid*, `en.wikipedia.org/wiki/Platonic_solid`.

[66] Zeno Rogue, *HyperRogue*, `roguetemple.com/z/hyper`.

[67] Zeno Rogue, *HyperRogue - Experiments with geometry*, `roguetemple.com/z/hyper/geoms.php`.

[68] Zeno Rogue, *HyperRogue - Models of hyperbolic geometry*, `roguetemple.com/z/hyper/models.php`.

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